Assignment 3.

This homework is due *Thursday*, September 20.

There are total 40 points in this assignment. 31 points is considered 100%. If you go over 31 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.1–2.3 in Bartle–Sherbert.

1. Quick cheat-sheet

Reminder. (Subsection 2.1.5 in textbook) Let $\mathbb{A}$ be a set with two operations $+$ and $\cdot$ satisfying A1–A4, M1–M3 and D (for example, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). The set $\mathbb{P} \subset \mathbb{A}$ is called the set of *positive elements* if

(i) If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$,
(ii) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$,
(iii) If $a \in \mathbb{A}$, then exactly one of the following holds: $a \in \mathbb{P}$, $a = 0$, $-a \in \mathbb{P}$.

Then $a < b$ if and only if $b - a \in \mathbb{P}$; $a \leq b$ if and only if $b - a \in \mathbb{P} \cup 0$.

One can prove the following (Theorem 2.1.7 in textbook):

Theorem. Let $a, b, c \in \mathbb{A}$.

(a) If $a > b$ and $b > c$ then $a > c$,
(b) if $a > b$, then $a + c > b + c$,
(c) if $a > b$, $c > 0$, then $ca > cb$,
if $a > b$, $c < 0$, then $ca < cb$.

2. Exercises

(1) (a) [2pt] ($\sim$Ex. 2.1.8a) Let x, y be rational numbers. Prove that $xy, x - y$ are rational numbers.

(b) [2pt] (Ex. 2.1.8b) Let x be a rational number, y an irrational number. Prove that $x + y$ is irrational. Prove that if, additionally, $x \neq 0$, then xy is irrational.

(c) [2pt] Let x, y be irrational numbers. Is it true that xy is always irrational? Is it true that xy is always rational?

(2) [2pt] Prove that there does not exist a rational number r such that $r^2 = 3$.

(3) (2.1.10) For $a, b, c, d \in \mathbb{R}$, prove that

(a) [2pt] if $a < b$, $c \leq d$, then $a + c < b + d$,

(b) [2pt] if $0 < a < b$, $0 < c \leq d$, then $0 < ac < bd$,

Every inequality you write should be accompanied by a reference to the exact property (i)–(iii), Theorem above or a previously proved claim that you are using.

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(4) [2pt] (2.1.12+) Let $a, b, c, d \in \mathbb{R}$ satisfy $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, an example where $ac > bd$, and an example where $ac = bd$.

(5) [3pt] (2.1.19) Prove that $\left(\frac{1}{2}(a+b)\right)^2 \le \frac{1}{2}\left(a^2+b^2\right)$ for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.

(6) In each case below, determine if P is a set of positive elements (i.e. whether it satisfies (i), (ii) and (iii)).
  (a) [2pt] $\mathbb{A} = \mathbb{Z}$, $P = \mathbb{N}$,
  (b) [2pt] $\mathbb{A} = \mathbb{Z}$, $P = -\mathbb{N}$,
  (c) [2pt] $\mathbb{A} = \mathbb{Q}$, $P = \{r \in \mathbb{Q} : \ r > 1\}$,
  (d) [2pt] $\mathbb{A} = \mathbb{C}$, $P = \{z = x + iy \in \mathbb{C} : \ x > 0\}$,
  (e) [3pt] Prove that for $\mathbb{A} = \mathbb{C}$, there is no set of positive elements. (In other words, one cannot imbue $\mathbb{C}$ with a meaningful order.)

  Note: items 6d, 6e deal with complex numbers that not everyone may be familiar with. So, these two questions are excluded from denominator of the grade for this homework. They are nevertheless included in numerator and you are encouraged to attempt them.

(7) ($\sim$Ex. 2.2.14bd,15bd) Determine and sketch the set of pairs (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy
  (a) [2pt] $|x| + |y| = 2$.
  (b) [2pt] $|x| - |y| = 2$.
  (c) [2pt] $|x| + |y| \ge 2$.
  (d) [2pt] $|x| - |y| \ge 2$.

(8) (a) [2pt] Let $S \subset \mathbb{R}$ be a bounded set. Let $S' \subset S$ be its nonempty subset. Show that $\sup S' \le \sup S$.
  (b) [2pt] (Ex. 2.3.10) Show that if A and B are bounded nonempty subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set and $\sup A \cup B = \sup\{\sup A, \sup B\}$.